JOURNAL OF APPROXIMATION THEORY 7, 132–142 (1973)

A Geometrical Approach to Property (SAIN)

RICHARD HOLMES* AND JOSEPH LAMBERT†

Department of Mathematics, Purdue University, Lafayette, Indiana 47907

Communicated by Oved Shisha

Received August 3, 1970

INTRODUCTION

In a recent paper [1], Deutsch and Morris introduced an abstract approximation-theoretic concept which they called "simultaneous approximation and interpolation which is norm-preserving" (SAIN). This concept, which was motivated by earlier work of Yamabe and Wolibner (see [1] for references), may be formulated in general normed linear spaces as follows.

DEFINITION. Let X be a normed linear space, M a dense subset of X, and Γ a finite dimensional subspace of X^*. The triple (X, M, Γ) has property (SAIN) if, for every $x \in X$ and $\epsilon > 0$, there exists $y \in M$ such that

$$\| x - y \| < \epsilon, \qquad \| x \| = \| y \|, \qquad \text{and} \qquad \gamma(x) = \gamma(y),$$

for every $\gamma \in \Gamma$.

As it stands this property pertains to constrained *dense* approximation, rather than to *best* approximation. Nevertheless, by restricting our attention to the case where M is (dense and) convex, we propose to approach the matter from the latter point of view. To this end, we introduce the (finite codimensional) subspace ${}^{\perp}\Gamma \subset X$, and consider the best approximation problem connected with this subspace. In many spaces X of interest, ${}^{\perp}\Gamma$ is a Chebyshev subspace; however, we need only assume that it has the *EF*-property of Morris et al. [2, 3]. From this assumption we obtain one of our main results (Theorem 1) which provides a pair of necessary and sufficient conditions for (X, M, Γ) to have property (SAIN).

In the original formulation [1], the subspace Γ was prescribed by a basis $\{\gamma_1, ..., \gamma_n\}$. In this setting Deutsch and Morris gave the following necessary condition for property (SAIN) [1, Theorem 2.3].

* Partially supported by the National Science Foundation and the Purdue Research Foundation.

† Partially supported by NDEA IV.

Deutsch–Morris Necessary Condition

The triple (X, M, Γ) has property (SAIN) only if each γ_i attains its norm solely at points of M, or not at all.

They also raised the question [1, Remark 2.6] asking when this last property is also a sufficient condition for property (SAIN). In Theorem 2 below we resolve this question as follows. Suppose that X is reflexive and rotund. Let T denote the norm duality map on X. Then for a fixed dense linear subspace M in X *the Deutsch–Morris necessary condition is equivalent to property* (SAIN) *for all* Γ *iff* $T(M)$ *is a linear subspace of* X^*. Since T is easily computed when X is an $L^p(\mu)$ space (Lemma 4), this result has an immediate application to a large number of (SAIN) problems in such spaces. It also leads to a new characterization of Hilbert spaces in terms of property (SAIN) (Theorem 3).

This paper consists of two main sections. The first presents several necessary and/or sufficient conditions for property (SAIN) in general normed spaces, some of which were mentioned above. We feel that these conditions establish a unified geometrical approach to property (SAIN). The second section contains numerous examples and applications of the theory, primarily for the case where $X = L^p(\mu)$ $(1 \leqslant p < \infty$, μ a positive measure), which we feel illustrate the efficacy of the approach developed in Section 1.

1. Property (SAIN) in Abstract Normed Spaces

Throughout this section we use the following notation: X is a real normed linear space and Θ is its zero vector; M is a dense convex subset of X, X^* is the continuous dual space of X, Γ is a finite dimensional subspace of X^*; $U(X)$ and $S(X)$ are respectively the closed unit ball and its boundary in X, and $L = {}^{\perp}\Gamma$ is the annihilator of Γ in X. For each $x \in X$, the set $P_L(x) = \{y \in L : \| x - y \| = \operatorname{dist}(x, L)\}$ is the set of *best approximations to* x *from* L; for some x it may be void. The mapping P_L from X into the closed bounded convex subsets of L is the *metric projection* of X on L. We write $x \perp L$ if $\Theta \in P_L(x)$, and say that the set $L^{\Theta} = \{x \in X : x \perp L\}$ is the *metric complement* of L in X. It is easily verified that the sets $x - P_L(x)$ and $\| x \| S(X) \cap (x + L)$ are the same whenever $x \in L^{\Theta}$; for such x we denote this set by G_x.

Lemma 1. *The triple* (X, M, Γ) *has property* (SAIN) *iff whenever* $x \in L^{\Theta}$ *we have*

$$G_x = \overline{M \cap G_x}.$$

(We note that this condition entails $\Theta \in M$, by definition of property (SAIN) and the fact that $G_{\Theta} = \{\Theta\}$.)

Proof. Suppose that (X, M, Γ) has property (SAIN) and that $y \in G_x$. Then $y = x - z$ for some $z \in L$ and $\| y \| = \| x \|$. Given $\epsilon > 0$, we want an $m \in M \cap G_x$ such that $\| y - m \| < \epsilon$. But property (SAIN) implies the existence of $m \in M$ with $\| y - m \| < \epsilon$, $\| m \| = \| y \|$, and $y - m = z_1 \in L$. However, this m is actually in G_x, since $m = y - z_1 = x - z - z_1 \in x + L$.

Conversely, assume the stated condition on G_x for $x \in L^{\ominus}$; let $y \in X$ and $\epsilon > 0$. We distinguish two cases.

Case a. $y \in L^{\ominus}$. By hypothesis, there exists $m \in M \cap G_y$ with $\| y - m \| < \epsilon$. By definition of G_y we also have $\| m \| = \| y \|$ and $m \in y + L$.

Case b. $y \notin L^{\ominus}$. In this case, $\operatorname{dist}(y, L) < \| y \|$, hence $y + L$ intersects $\| y \| \mathring{U}(X)$. Since $\operatorname{codim} L < \infty$, the Singer–Yamabe theorem (e.g., [1, Theorem 1.1]) implies $M \cap (y + L)$ is dense in $y + L$. In particular, there exists $m \in M \cap (y + L)$ such that $\| y - m \| < \epsilon$ and $\| m \| < \| y \|$. The proof is now completed by an appeal to [1, Lemma 2.3].

Remark. It is possible to prove Lemma 1 by carefully following the argument used to establish the theorem of McLaughlin and Zaretzki [7, p. 56]. We prefer the proof given above, however, because of its simplicity and brevity. The brevity was made possible by an appropriate utilization of the lemmas of [1, pp. 357–358].

We note that $G_x \subset L^{\ominus}$, that the norm is constant on G_x, and that G_x is closed and convex. The definitions of G_x and $P_L(x)$ do not involve the dense set M. Thus, Lemma 1 provides our first geometric characterization of property (SAIN) by showing this property to be equivalent to the density of M in certain convex sets defined by L. We should also point out that the sets G_x, although contained in the spheres $\| x \| S(X)$, are generally not *faces* of the corresponding balls, because they may fail to be extremal subsets of these balls.

COROLLARY 1. *The condition $L^{\ominus} \subset M$ always implies property* (SAIN).

As we show later (Theorem 1), the condition of Corollary 1 may in fact be equivalent to property (SAIN), given additional information about L and M. Meanwhile, the next result shows that a weakened form of this condition is always necessary for property (SAIN).

LEMMA 2. *The following two conditions are equivalent to each other and are implied by property* (SAIN):

(a) $L^{\ominus} = \overline{M \cap L^{\ominus}}$;

(b) $S(X) \cap L^{\ominus} = \overline{M \cap S(X) \cap L^{\ominus}}$.

Further, these conditions imply property (SAIN) *in the special case where* $\dim \Gamma = 1$.

Proof. We omit the (routine) proof that (a) and (b) are necessary for property (SAIN). That (a) and (b) are equivalent follows from the fact that L^{Θ} is closed under scalar multiplication. Finally, suppose that $\Gamma = \operatorname{span}\{\gamma\}$, that (b) holds, that $x \perp L$, that $\| x \| = 1$, that $y \in G_x$, and that ϵ in $(0, 1)$ is given. Then (b) implies the existence of $m \in M \cap S(X) \cap L^{\Theta}$ such that $\| m - y \| < \epsilon$. We show that $m - y \in L$ and conclude by use of Lemma 1. We may assume that $\| \gamma \| = 1$. Now,

$$1 = \| y \| = \operatorname{dist}(y, L) = \sup\{f(y) : f \in S(\Gamma)\},$$

and similarly for m. Thus $| \gamma(m)| = | \gamma(y)| = 1$, and since

$$1 > \epsilon > \| m - y \| \geqslant | \gamma(m - y)|,$$

we must have $\gamma(m) = \gamma(y)$ or $m - y \in L$. Q.E.D.

We remark that it is an open question whether the above conditions (a) and (b) are equivalent to property (SAIN) when $\dim \Gamma > 1$.

Before stating the next lemma, which provides one more necessary condition of geometrical type for property (SAIN), we make the following definitions.

DEFINITION. Let $x \in X$, $x \neq \Theta$. The *conjugate set* for x is

$$\partial(x) = \{f \in S(X^*) : f(x) = \| x \|\}.$$

The *contact set* for x is $F_x = \| x \| S(X) \cap \bigcap \{f^{-1}(\| x \|) : f \in \partial(x)\}$.

The conjugate set for x, being just the subdifferential of the norm at x, is always w^*-compact, convex, and nonempty. The contact set for x, being the intersection of a multiple of the unit ball of X and its hyperplanes of support at x, is a closed, convex and nonempty face of $\| x \| U(X)$. Its definition is evidently independent of the subspaces L and M. However, we note that $x \perp L \Leftrightarrow \partial(x) \cap \Gamma \neq \varnothing \Leftrightarrow F_x \subset L^{\Theta}$ for *any* subspace $L = {}^{\perp}\Gamma$, and that if $x \perp L$ and $\partial(x) \subset \Gamma$ (which would be the case if x were a smooth point of X) then $G_x \subset F_x$.

LEMMA 3. *The following condition is implied by property* (SAIN): *whenever* $\partial(x) \subset \Gamma$ *we have*

$$F_x = \overline{M \cap F_x}.$$

The proof is similar to previous proofs and is, therefore, omitted.

The next result shows that for important special classes of subspaces Γ and convex sets M, a strong form of the Deutsch–Morris necessary condition (cf. Introduction) and the sufficient condition of Corollary 1 are each equivalent to property (SAIN).

DEFINITION. L is an *EF-subspace* of X if, for every $x \in X$, the set $P_L(x)$ is nonempty and finite dimensional. M is *affine* if u, $v \in M$ implies $tu + (1 - t)\,v \in M$ for all real t.

The *EF*-subspaces played a role in [3] and were formally defined by Morris in [2, p. 800].

THEOREM 1. *Suppose that L is an EF-subspace of X and that M is (dense and) affine. Then the following statements are equivalent*:

(a) (X, M, Γ) *has property* (SAIN);

(b) *each nonzero* $\gamma \in \Gamma$ *attains its norm solely on* M, *or not at all*;

(c) $L^\Theta \subset M$.

Proof. It is clear that conditions (b) and (c) are equivalent for any L. We already know (Corollary 1) that (c) $\Rightarrow$ (a). We now complete the proof by showing (a) $\Rightarrow$ (c). Let $x \in L^\Theta$. According to Lemma 1, $M \cap G_x$ is dense in G_x. Since M is affine it follows that $M \cap \operatorname{aff} G_x$ is dense in $\operatorname{aff} G_x$, where "aff" means "affine hull of." But $\operatorname{aff} G_x$ is a finite dimensional linear variety and $M \cap \operatorname{aff} G_x$ is a linear subvariety. It follows that the two are equal, that is, $x \in G_x \subset \operatorname{aff} G_x \subset M$. Q.E.D.

We might note that the conditions of Theorem 1 actually imply that M must be a linear subspace of X, since, as was noted in Lemma 1, property (SAIN) implies $\Theta \in M$.

DEFINITION. Let X be a normed linear space. The *norm duality map* T is the point-to-set mapping of X into the w^*-compact convex subsets of X^* given by

$$T(\Theta) = \Theta,$$
$$T(x) = \|x\|\,\partial(x), \qquad \text{if} \quad x \neq \Theta.$$

This mapping has been studied by Browder [4] and Cudia [5]. We note that, for $x \neq \Theta$, $T(x)$ is a singleton exactly when x is a smooth point of X (written $x \in \operatorname{sm}(X)$), and that the sets $T(x)$, $T(y)$ are disjoint for every $x, y \in X$ exactly when X is rotund. Further, $T(X) = X^*$ exactly when X is reflexive, although $T(X)$ is dense in X^* in any case when X is complete. In particular, when X is reflexive, rotund, and smooth, then T is a bijection from X to X^*.

COROLLARY 2. *Suppose that* $\Gamma \subset T(M)$ *and that* $S(\Gamma) \subset \mathrm{sm}(X^*)$. *Then* (X, M, Γ) *has property* (SAIN).

Proof. We show that $L^{\ominus} \subset M$. Let $x \in L^{\ominus}$. Then there is $f \in \Gamma \cap T(x)$, hence $f \in T(M)$. It follows that $x \in M$, for otherwise f would attain its norm nonuniquely and so would not be a smooth point of X^*.

We recall [1] that property (SAIN) was originally formulated in terms of a basis $\{\gamma_1, ..., \gamma_n\}$ for Γ. Deutsch and Morris gave a necessary condition (cf. Introduction) for property (SAIN) in this setting, and showed [1, Corollary 2.3, Theorems 3.2 and 5.1] that in certain special cases this condition was equivalent to property (SAIN). They also raised the general question of the sufficiency of this condition. Our next theorem provides an answer to this question for a special class of spaces X.

THEOREM 2. *Suppose that* X *is reflexive and rotund and that* M *is a (dense) linear subspace of* X. *Then the Deutsch–Morris necessary condition is always sufficient for* (X, M, Γ) *to have property* (SAIN) *iff* $T(M)$ *is a linear subspace of* X^*.

Proof. Suppose that $T(M)$ is a linear subspace of X^* and that each γ_i attains its norm (uniquely) on M. Now X^* is smooth (because X is reflexive and rotund) and $\mathrm{span}\{\gamma_1, ..., \gamma_n\} = \Gamma \subset T(M)$, since each $\gamma_i \in T(M)$. Hence, property (SAIN) results from Corollary 2. Conversely, suppose that $T(M)$ is not a linear subspace of X^*. Since $T(cx) = cT(x)$ for all real c, there must exist m_i ($i = 1, 2$) in M and $\gamma_i \in \partial(m_i)$ such that $\gamma_1 + \gamma_2$ attains its norm (uniquely) at $y \in X \backslash M$. Let $\Gamma = \mathrm{span}\{\gamma_i\}$ and $L = {}^{\perp}\Gamma$, so that codim $L = 2$ in X. We have $y \in L^{\ominus}$ and since L is a Chebyshev subspace, condition (c) of Theorem 1 fails. Thus $(X, M, \mathrm{span}\{\gamma_1, \gamma_2\})$ does not have property (SAIN), although the Deutsch–Morris necessary condition is in force. Q.E.D.

When X is a Hilbert space, the norm duality map T is linear; in fact, modulo the Riesz representation theorem, it is just the identity map. It is, therefore, an immediate consequence of Theorem 2 that the Deutsch–Morris necessary condition is always equivalent to property (SAIN) for dense subspaces of Hilbert spaces—a fact already established somewhat differently in [1, Theorem 3.1]. Our final result of this section gives a converse implication and thereby provides a characterization of Hilbert spaces in terms of property (SAIN).

THEOREM 3. *Let* X *be reflexive and rotund, and suppose that the Deutsch–Morris necessary condition is always equivalent to property* (SAIN). *Then* X *is a Hilbert space.*

Proof. In view of the preceding remarks, it will suffice to show that if X is not a Hilbert space then there exists a dense linear subspace $M \subset X$ for which $T(M)$ is not a subspace of X^*. For we may then apply Theorem 2 to obtain a contradiction to our stated hypothesis. But if X is not a Hilbert space then neither is X^*. Hence, in X^* orthogonality is not left additive [6, Theorem 2], so there exist functionals $\varphi, \gamma_i, \gamma_2 \in X^*$ for which $\gamma_i \perp \operatorname{span}\{\varphi\}$ but $\gamma_1 + \gamma_2 \not\perp \operatorname{span}\{\varphi\}$. Let $H = \{x \in X : \varphi(x) = 0\}$. Now $\gamma_i \in T(H)$ since $\gamma_i \perp H^\perp$ and $(H^\perp)^\perp = H$ because X is reflexive. The same argument implies $\gamma_1 + \gamma_2 \notin T(H)$. Suppose that $\gamma_1 + \gamma_2 \in T(x)$ for (a unique) $x \in X \backslash H$. Let P be a one-dimensional subspace of X which is disjoint from both x and H. Let Q be a dense (and proper) subspace of H which contains the two points where the γ_i attain their norms. Finally let $M = P \oplus Q$. Then $\overline{M} = X$ but but $T(M)$ is not linear. Q.E.D.

2. Applications of the Preceding Theory

We begin by taking $X = l^1$, and M the subspace of vectors with only finitely many nonzero coordinates. In [1, Corollary 6.2] it was shown that (X, M, Γ) had property (SAIN) provided that the elements of Γ were "eventually constant." The present authors conjecture that in fact (X, M, Γ) has property (SAIN) *for all* $\Gamma \subset X^* = l^\infty$. (*Note added in proof.* This conjecture has been substantiated. See J. M. Lambert, Simultaneous approximation and interpolation in l^1, *Proc. Amer. Math. Soc.* **32** (1972), 150–152.) However, only the following special case of this conjecture has been established.

COROLLARY 3. *If $\Gamma \subset c_0$ (the pre-dual of l^1) then (l^1, M, Γ) has property* (SAIN).

Proof. According to Corollary 1, it is sufficient to show $L^\Theta \subset M$. Let $x \in L^\Theta$. There exists $\gamma \in \Gamma \cap T(x)$, and so

$$\sum_{n=1}^{\infty} \gamma(n)\, x(n) = \| \gamma \|_\infty \sum_{n=1}^{\infty} | x(n) | .$$

Here $\gamma(n)$ is the nth component of γ, etc. This equation requires

$$(\gamma(n)/\| \gamma \|_\infty)\, x(n) = | x(n) |,$$

for every n. But $\gamma \in c_0$, so that $| \gamma(n) | / \| \gamma \|_\infty < 1$ for sufficiently large n. For all such n we therefore have $x(n) = 0$, that is, $x \in M$. Q.E.D.

It is interesting to note that M is the smallest subspace of l^1 for which the statement of Corollary 3 is true. This fact follows from the next theorem. The proof, which depends on Lemma 2 and the following definition, is omitted.

DEFINITION. x is an *exposed point* of a convex set $K \subset X$ (written $x \in \exp(K)$) if there is a hyperplane of support to K which touches K only at x. If $X = Y^*$ is a dual space, then x is a *regularly exposed point* of $K \subset X$ (written $x \in \operatorname{reg}\exp(K)$) if $x \in \exp(K)$ and the associated hyperplane is defined by an element of Y (qua element of Y^{**}).

THEOREM 4. *If* (X, M, Γ) *has property* (SAIN) *for all* $\Gamma \subset X^*$ *then* $M \supset \exp(U(X))$. *If* $X = Y^*$ *and* (X, M, Γ) *has property* (SAIN) for all $\Gamma \subset Y$, *then* $M \supset \operatorname{reg}\exp(U(X))$.

We note that when $X = l^1$, $\exp(U(X) = \operatorname{reg}\exp U(X) = \{\pm e_m\}$, where $e_m(n) = \delta_{mn}$. Thus the subspace M of Corollary 3 is exactly $\operatorname{span}\{\operatorname{reg}\exp(U(l_1))\}$.

We now indicate some further applications of Lemma 2. Suppose $X = L^1(\mu)$ where the measure is such as to guarantee that $X^* = L^\infty(\mu)$. Let M be a subspace of X which contains the simple functions. For example, M could be the space of simple functions itself or the (generally larger) space of $L^1(\mu)$ functions, each of which vanishes off a set of finite measure. For a given $f \in L^\infty(\mu)$ we consider the problem of (SAIN) for $(X, M, \{f\})$. We distinguish two cases.

Case a. $\mu(\{t : |f(t)| = \|f\|_\infty\}) = 0$. In this case, $L^\Theta = \{\Theta\}$, where $L = {}^\perp\operatorname{span}\{f\}$. Hence property (SAIN) is immediate from Corollary 1.

Case b. $\mu(\{t : |f(t)| = \|f\|_\infty\}) = \mu(A) > 0$. Now we have

$$L^\Theta = \{x \in L^1(\mu) : x \text{ vanishes a.e. } [\mu] \text{ outside of } A\}.$$

Evidently, the simple functions which vanish outside A are dense in L^Θ, so the conditions of Lemma 2 are met and property (SAIN) follows.

We summarize these observations in the next corollary, after remarking that the case where M consists of functions each vanishing off a set of finite measure was given a direct ad hoc proof in [1, Theorem 6.1].

COROLLARY 4. *When X is an L^1 space whose dual is the corresponding L^∞ space, and M contains the simple functions, then (X, M, Γ) has property* (SAIN) *for all one dimensional $\Gamma \subset X^*$.*

Clearly, the preceding argument can be used to obtain the conclusion of Corollary 4 for other dense subsets M if we assume more structure in the underlying measure space. Thus, if μ is a Borel measure on a locally compact Hausdorff space, we can replace M by the space of continuous functions with compact support. Similar examples will suggest themselves to the interested reader.

Finally, we consider some applications of Theorem 2 and Corollary 2. We let $X = L^p(\mu)$ for $1 < p < \infty$ and some positive measure μ. As usual $X^* = L^q(\mu)$ where $q = p/(p-1)$. We denote the norm duality map on $L^p(\mu)$ by T_p. The next lemma provides some pertinent information about T_p.

LEMMA 4. (a) *T_p is a homeomorphism from $L^p(\mu)$ onto $L^q(\mu)$ and $T_p^{-1} = T_q$;*

(b) *if $x \in L^p(\mu)$, then $T_p(x) = |x(\cdot)|^{p-1} \operatorname{sgn} x(\cdot)/\|x\|^{p-2}$;*

(c) *if $M \subset \bigcap_{p>1} L^p(\mu)$ and if $T_p(M) \subset M$ for all $p > 1$, then $T_p(M) = M$ for all such p.*

We are now going to consider several special examples of measures μ and dense linear subspaces M. Our primary concern in these examples is to decide whether or not the Deutsch–Morris necessary condition is sufficient for $(X, M, \operatorname{span}\{\gamma_1, ..., \gamma_n\})$ to have property (SAIN). Theorem 2 allows us to consider instead the equivalent question of whether or not $T_p(M)$ is linear in X^*.

First let μ be arbitrary and let M consist of either the simple functions or the functions vanishing off a set of finite measure. Then clearly Lemma 4 (parts (b) and (c)) shows that $T_p(M) = M$. (A proof from basic principles of the sufficiency when M consists of the functions vanishing off sets of finite measure was given in [1, Theorem 5.1].) If μ is also assumed finite and we take $M = L^\infty(\mu)$, then the same argument shows $T_p(M) = M$. Finally, if μ is a finite regular Borel measure and M consists of the bounded continuous functions, the continuous functions which vanish at infinity, or the continuous functions with compact support, then once again we have $T_p(M) = M$. Thus in all these cases the Deutsch–Morris necessary condition is equivalent to property (SAIN).

Next we continue with μ as in the preceding example and require that $p \neq 2$. We are going to see that if M consists of any of the standard spaces of *smooth* functions, then M is too "sparse" for $T_p(M)$ to be linear. Thus it will follow from Theorem 2 that when M is one of the indicated spaces of smooth functions there will exist $\gamma_1, ..., \gamma_n$ each of which attains its norm (uniquely) on M but for which $(L^p(\mu), M, \operatorname{span}\{\gamma_1, ..., \gamma_n\})$ does not have property (SAIN). In fact, such a triple will have property (SAIN) exactly when $\operatorname{span}\{\gamma_1, ..., \gamma_n\} \subset T_p(M)$. In the particular case where μ is Lebesgue measure on some interval and M is the space of polynomials, the set $T_p(M)$ contains *no* two-dimensional subspaces; hence for (X, M, Γ) to have property (SAIN) it is necessary that $\dim \Gamma = 1$. We may contrast this observation with the following corollary to Theorem 1, which shows that, given X and M, it is generally to be expected that (X, M, Γ) will have property (SAIN) for some one-dimensional subspaces Γ.

COROLLARY 5. *If X is a normed linear space and M a dense convex subset of X which contains an exposed point of $U(X)$, then there exists $\gamma \in X^*$ such that $(X, M, \mathrm{span}\{\gamma\})$ has property* (SAIN).

In the next two examples we let μ be Lebesgue measure on $[0, 1]$, and for $0 < \alpha \leqslant 1$, we write $\mathrm{Lip}(\alpha)$ for the space of functions satisfying a Lipschitz condition with exponent α.

EXAMPLE 1. Let $1 < p < 2$, $0 < \alpha \leqslant 1$, $x(t) = t^\alpha$, and $y(t) = 1$ for $0 \leqslant t \leqslant 1$. We use Lemma 4(a) to solve the equation $T_p(x) + T_p(y) = T_p(z)$ for z. It results that

$$z(t) = B(1 + At^{\alpha(p-1)})^{1/(p-1)},$$

where A and B are positive constants. Applying the mean-value theorem to the difference $z(t) - z(0)$, we obtain

$$\begin{aligned} t^{-\alpha}(z(t) - z(0)) &= AB^\alpha(\alpha t^{1-\alpha})(1 + Ar^{\alpha(p-1)})^{(2-p)/(p-1)}\, r^{\alpha p-\alpha-1} \\ &\geqslant Ct^{1-\alpha}/r^{1+\alpha-\alpha p} > Ct^{\alpha p-2\alpha}. \end{aligned}$$

Here C is a positive constant and $0 < r < t$. Since $2 > p$ this term is unbounded as $t \to 0+$. That is, $x, y \in \mathrm{Lip}(\alpha)$ but $T_p(x) + T_p(y) \notin T_p(\mathrm{Lip}(\alpha))$, hence this latter set is not linear. We note that when $\alpha = 1$, the example also shows that $T_p(C^k)$ is not linear for $1 \leqslant k \leqslant \infty$, nor is T_p ({polynomials}).

EXAMPLE 2. *Let* $2 < p < \infty$, $0 < \alpha \leqslant 1$, $x(t) = 1 + t^\alpha$ and $y(t) = \lambda$, a positive constant to be determined later. As before we compute z from the equation $T_p(x) - T_p(y) = T_p(z)$, and find

$$z(t) = B(A(1 + t^\alpha)^{p-1} - \lambda)^{1/(p-1)},$$

where A and B are positive constants. We choose λ so that $z(0) = 0$. Now

$$t^{-\alpha}z(t) = B([A(1 + t^\alpha)^{p-1} - \lambda]/t^{\alpha(p-1)})^{1/(p-1)},$$

so by L'Hospital's rule,

$$\lim_{t\to 0+} t^{-\alpha}z(t) = \lim_{t\to 0+} Ct^{\alpha-1-\alpha p+\alpha+1} = \lim_{t\to 0+} Ct^{2\alpha-\alpha p},$$

where C is a positive constant. As $2 < p$, this last exponent is negative so that $z \notin \mathrm{Lip}(\alpha)$. Now the remarks made at the end of example 1 apply.

We might finally remark that both the above examples can be considerably generalized. For example, we may replace μ by any finite positive regular Borel measure, since this only changes the values $T_p(x)$ by a positive constant. We may also replace $[0, 1]$ by a metric space (Ω, d), provided there is a cluster point $t_0 \in \Omega$. If so, we replace t^α in the preceding examples by $\min(1, d(t, t_0)^\alpha)$.

Note added in proof (January 11, 1973). Since this paper was submitted, a research announcement has appeared in the Russian literature [V. Shmatkov, On simultaneous approximation and interpolation in Banach spaces, *Dokl. Akad. Nauk Armyanskoi SSR* **53** (1971), 65–70], which to some extent overlaps Part I of the present paper and also the McLaughlin–Zaretzki paper [7]. In particular, a result equivalent to our Theorem 3 is announced.

References

1. F. Deutsch and P. Morris, On simultaneous approximation and interpolation which preserves the norm, *J. Approximation Theory* **2** (1969), 355–373.
2. P. Morris, Metric projections onto subspaces of finite codimension, *Duke Math. J.* **35** (1968), 799–808.
3. J. Blatter, P. Morris, and D. Wulbert, Continuity of the set-valued metric projection, *Math. Ann.* **178** (1968), 12–24.
4. F. Browder, Multivalued monotone nonlinear mappings and duality mappings in Banach spaces, *Trans. Amer. Math. Soc.* **118** (1965), 338–351.
5. D. Cudia, The geometry of Banach spaces: Smoothness, *Trans. Amer. Math. Soc.* **110** (1964), 284–314.
6. R. James, Inner products in normed linear spaces, *Bull. Amer. Math. Soc.* **53** (1947), 559–566.
7. H. McLaughlin and P. Zaretzki, Simultaneous approximation and interpolation with norm preservation, *J. Approximation Theory* **4** (1971), 54–58.